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PROPERTIES OF RESIDUAL MIXING DISTRIBUTIONS RESULTING FROM ARBITRARY MIXTURES OF EXPONENTIAL LIFE DISTRIBUTIONS

David Russell Campbell

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PROPERTIES OF RESIDUAL MIXING DISTRIBUTIONS RESULTING FROM ARBITRARY MIXTURES OF EXPONENTIAL LIFE DISTRIBUTIONS

by

David Russell Campbell

June 1977

Thesis Advisor: James D. Esary

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A mixture of failure rates can be present in an apparently homogeneous population of "devices" due to variability either in their manufacture or in the severity of their service environments. An initial mixing distribution is the probability distribution for different failure rates in such a population. This distribution may be updated to yield its related residual mixing distribution, which is the probability distribution for

different failure rates in the population of survivors after a specified period of service or "burn-in." Residual mixing distributions resulting from arbitrary mixtures of constant failure rates are shown to be stochastically ordered (decreasingly) as the period of service or burn-in is increased, and to approach in the limit a distribution degenerate at the smallest failure rate "present" in the population. Properties of expected value ordering, stochastic ordering, failure rate ordering and likelihood ratio ordering are investigated to show that, of these, only likelihood ratio ordering between two initial mixing distributions is sufficient to guarantee an ordering between the expected values of their respective residual mixing distributions over time.

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Properties of Residual Mixing Distributions Resulting from Arbitrary Mixtures of Exponential Life Distributions

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ABSTRACT

A mixture of failure rates can be present in an apparently homogeneous population of "devices" due to variability either in their manufacture or in the severity of their service environments. An initial mixing distribution is the probability distribution for different failure rates in such a population. This distribution may be updated to yield its related residual mixing distribution, which is the probability distribution for different failure rates in the population of survivors after a specified period of service or "burn-in." Residual mixing distributions resulting from arbitrary mixtures of constant failure rates are shown to be stochastically ordered (decreasingly) as the period of service or burn-in is increased, and to approach in the limit a distribution degenerate at the smallest failure rate "present" in the population. Properties of expected value ordering, stochastic ordering, failure rate ordering and likelihood ratio ordering are investigated to show that, of these, only likelihood ratio ordering between two initial mixing distributions is sufficient to guarantee an ordering between the expected values of their respective residual mixing distributions over time.

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SUMMARY

It is well-known that the failure rate function for arbitrary mixtures of exponential life distributions decreases over time, approaching as a limit the smallest failure rate positively present in the mixture.

It is shown here that the sequence of residual mixing distributions which describes the distribution of failure rates present in the mixture of survivors is not only decreasing in expectation over time, but is also a stochastically decreasing sequence, degenerate in the limit at the smallest failure rate present in the mixture. Accordingly, for any particular value, the proportion of items present in the population of survivors having failure rates greater than that value decreases over time. This strengthens the notion that the shorter-lived members of the population are being "weeded out."

Should it be possible to alter the initial distribution of failure rates present in the mixture, one would seek an alteration which would be better than the original mixture, not only initially, but over time as well. A reasonable way to measure "better" may be found in the failure rate function of the mixture. That is, one might seek an alteration to the original mixing distribution which would lead to a reduction in the mixture failure rate function over time.

In seeking to find conditions on the alteration sufficient to guarantee the above-described improvement, several orderings among the initial mixing distributions are considered. They are, listing from weakest to strongest, expected value ordering, stochastic ordering, failure rate ordering, and likelihood ratio ordering. The results of this investigation show that, of the orderings considered, only likelihood ratio ordering is sufficient to guarantee the desired improvement.

Likelihood ratio ordering is shown to be equivalent in the continuous case to distributions of Polya Type II. This result indicates that many of the more commonly encountered distributions belong to this class, with the gamma distribution being a convenient and relevant example. Gamma mixtures of exponentials are employed in reliability evaluation for the Trident Missile program.

It is shown that, in an n-point discrete mixture, to achieve the desired improvement by shifting mass from one failure rate to a smaller one, the mass must be removed from the largest failure rate and added to the smallest failure rate. In the use of this mixture to describe the procurement of items from n different sources, this result states that simply shifting mass from one source to a better one will lead to a mixture which will exhibit a lower initial failure rate, but a higher subsequent one. One must shift mass from the poorest source to the best source to prevent this subsequent crossover between mixture failure rates.

I. INTRODUCTION

A. BACKGROUND

Mixtures of probability distributions have intuitive appeal in a number of diverse applications of probabilistic models. Their use has spanned many years, dating back to before the turn of the century, for example, Karl Pearson's work in 1894 [1]. The breadth of their potential use was indicated by Feller [2] in saying that "every distribution may be represented as a mixture."

As indicated by Blischke [3], two types of problems associated with mixtures of distributions have received much attention in the literature. They are the problems of identifiability and parameter estimation.

The problem of identifiability is one of unique characterization. As defined by Teicher in [4], for $\mathcal{F} = \{F\}$ a family of distributions, a μ-mixture of $\mathcal{F}$, say H, will be called identifiable if for any probability measure μ^*, the relationship

$$H(x) = \int F(x)d\mu(F) = \int F(x)d\mu^*(F)$$

implies that $\mu = \mu^*$. Other investigations into the identifiability of mixtures are given in [5], [6] and [7].

Given whatever assumptions one is willing to make concerning the families of distributions which may be involved in the mixture, the specifying parameters of those distributions must generally be estimated from sample information. Cohen and

Falls [8] provide a good example of this sort of investigation. They examine estimation procedures in a number of different mixtures, including mixtures of two Poissons, two exponentials, and two Weibulls. Their specific application is in the analysis of atmospheric data. Mixtures of binomials are examined in [3] and [9], the latter with the field of advertising as an application. Parameter estimation for mixtures of exponentials is treated in [10] and [11].

This thesis treats general mixtures of exponentials. The recognition of such a mixture as a Laplace-Stieltjes transform guarantees its identifiability due to the uniqueness of determining functions of such transforms as asserted in [12] and [2]. The purpose of this effort is to investigate the properties of such a model with emphasis on the interpretation and application of results in the reliability context.

The results in chapter II are essentially those reported earlier by the author in [13].

B. THE MODEL

The model we shall consider is a mixture of the form

$$\begin{aligned} \bar{F}(t) &= \int_0^\infty e^{-\lambda t}\, dG(\lambda), \quad t \geq 0 \\ &= P(T > t), \end{aligned} \tag{1.1}$$

where $\bar{F}(t)$ is the survival function of the random variable T. $\bar{F}(t;\lambda) = e^{-\lambda t}$ is the survival function of the component distribution given that $\Lambda = \lambda$ for Λ a nonnegative random variable

having cumulative distribution function G. We can refer to such a mixture as a G-mixture of exponentials.

We take the usual definition for a failure rate function as given in [14], i.e., for the random variable T having density f(t) and survival function $\overline{F}(t)$, the failure rate function h(t) is defined by

$$h(t) = \frac{f(t)}{\overline{F}(t)}, \text{ for all } t \text{ such that } F(t) > 0.$$

The initial mixing distribution G may be modified to yield a related distribution indexed according to the value of $t \geq 0$, i.e., let $G_t(\lambda) = P(\Lambda \leq \lambda | T > t)$ and define $\Lambda(t)$ to be a random variable having cumulative distribution function G_t, which we recognize to be a conditional distribution for Λ, given that $T > t$. The relationship between a mixing distribution G and its corresponding distribution G_t is given by a straightforward application of Bayes' Theorem which yields

$$dG_t(\lambda) = \frac{e^{-\lambda t}}{E(e^{-\Lambda t})} dG(\lambda), \tag{1.2}$$

where the expectation is with respect to G, the distribution of Λ.

We shall find it necessary to consider alternative mixing random variables, for example, Λ_1 and Λ_2. We associate with each random variable Λ_i its distribution G_i and its corresponding conditional distribution $G_{i,t}$ as above.

C. THE RELIABILITY PERSPECTIVE

In the reliability context for the expression (1.1), T represents the time to failure of an item drawn at random from a population of items having different exponential life distributions, mixed according to G, the distribution of Λ.

The exponential distribution has been widely employed in modeling random lifetimes, due largely to its tractability and to the fact that it is the limiting life distribution of complex equipment as the complexity and time of operation increase. The assumption for the latter result is essentially that the lifetime of the equipment is a superposition of many renewal processes [14].

The failure rate of such equipment may be considered to be related to both quality of manufacture and service environment [15]. Thus, the consideration of both variability of manufacture and variability of service environment leads, individually and jointly, to the choice of a mixture of component distributions as the description of the life distribution of an item drawn at random from such a population. The use of the Stieltjes integral [12] admits continuous, discrete, or mixed distributions G.

The conditional distribution for Λ, given that $T > t$, G_t, is the residual mixing distribution at time t. It represents the revised distribution for the failure rate of an item randomly drawn from the mixture, given that the item is still functioning at time t. As an item continues to survive it admits a continual updating of the information at hand

concerning the mixture from which it was drawn. G_t also represents the relevant mixing distribution for an item randomly drawn from the population of survivors at time t.

For such a G-mixture of exponentials it is well known [16] that the mixture failure rate h(t) is decreasing in t (we shall use 'decreasing' for non-increasing and 'increasing' for non-decreasing), and that the mixture failure rate approaches as a limit, with increasing time t, the least of all parameter values present in the mixture [17].

Aldrich and Morton [18] and O-Bar [19] showed that at time t the failure rate function of a G-mixture of exponentials is the expected value of the residual mixing distribution corresponding to that time, i.e.,

$$h(t) = E[\Lambda(t)]. \tag{1.3}$$

Thus, the sequence of residual mixing distributions is decreasing in expectation. This fact supports the intuitive notion that with increasing time the shorter-lived members of the mixture are being "weeded out." O'Bar [19] also showed that the slope of the mixture failure rate function over time is related to the variance of the residual mixing distribution by

$$h'(t) = -VAR[\Lambda(t)].$$

D. EXAMPLES

1. Two-Point Discrete Mixture

This represents the simplest discrete case. Here, G is defined for $\lambda_1 < \lambda_2$ by

$$\Lambda = \begin{cases} \lambda_1, \text{ with probability } p_1 \\ \lambda_2, \text{ with probability } p_2. \end{cases}$$

This could be thought of as a two-vendor situation, or a single-vendor situation with two distinct service environments. As developed in [19], the mixture survival function is given by

$$\begin{aligned} \bar{F}(t) &= P(T > t) \\ &= P(\Lambda = \lambda_1)P(T > t|\Lambda = \lambda_1) + P(\Lambda = \lambda_2)P(T > t|\Lambda = \lambda_2) \\ &= p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}. \end{aligned}$$

The mixture failure rate function is

$$\begin{aligned} h(t) = \frac{f(t)}{\bar{F}(t)} &= \frac{p_1\lambda_1 e^{-\lambda_1 t} + p_2\lambda_2 e^{-\lambda_2 t}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}} \\ &= \frac{p_1\lambda_1 + p_2\lambda_2 e^{-(\lambda_2 - \lambda_1)t}}{p_1 + p_2 e^{-(\lambda_2 - \lambda_1)t}} \end{aligned} \tag{1.4}$$

The conditional probability that the item was drawn from component distribution 1, given that it is still functioning at time t, $p_1(t)$, is given by

$$\begin{aligned} p_1(t) &= P(\Lambda = \lambda_1|T > t) \\ &= \frac{P(\Lambda = \lambda_1, T > t)}{P(T > t)} = \frac{p(T > t|\Lambda = \lambda_1)P(\Lambda = \lambda_1)}{P(T > t)} \\ &= \frac{p_1 e^{-\lambda_1 t}}{p_1 e^{-\lambda_1 t} + p_2 e^{-\lambda_2 t}} = \frac{p_1}{p_1 + p_2 e^{-(\lambda_2 - \lambda_1)t}} \end{aligned}$$

The conditional probability that the item was drawn from component distribution 2, given survival to time t is

$$p_2(t) = 1 - p_1(t) = \frac{p_2}{p_2 + p_1 e^{(\lambda_2 - \lambda_1)t}}$$

$p_1(t)$ and $p_2(t)$ are called the residual mixing probabilities and can also be interpreted as the proportions of items with the component failure rates λ_1 and λ_2 in the population of survivors at time t. The residual mixing distribution G_t is then defined by

$$\Lambda(t) = \begin{cases} \lambda_1, \text{ with probability } p_1(t) \\ \lambda_2, \text{ with probability } p_2(t). \end{cases}$$

Note that $p_1(0) = p_1$ and $p_2(0) = p_2$. As t increases, $p_1(t)$ increases from p_1 to 1 and $p_2(t)$ decreases from p_2 to 0.

2. Gamma Mixture

This is a frequently cited continuous model, due primarily to its tractability. For example, see [15] and [20]. Here, Λ has the density

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \text{ for } \alpha, \beta, \lambda > 0.$$

The mixture survival function is

$$\bar{F}(t) = P(T > t) = \int_0^\infty e^{-\lambda t} g(\lambda)\, d\lambda$$

$$= \int_0^\infty e^{-\lambda t} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda$$

$$= \frac{\beta^\alpha}{(\beta + t)^\alpha} \int_0^\infty \frac{(\beta + t)^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-(\beta + t)\lambda} d\lambda$$

$$\bar{F}(t) = \left[\frac{\beta}{\beta + t}\right]^\alpha, \quad t \geq 0.$$

The mixture failure rate function is given by

$$h(t) = \frac{f(t)}{\bar{F}(t)} \tag{1.5}$$

$$= \frac{\alpha}{\beta + t}$$

The residual mixing density is

$$g_t(\lambda) = \frac{e^{-\lambda t} g(\lambda)}{\bar{F}(t)}$$

$$= \left[\frac{\beta + t}{\beta}\right]^\alpha \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-(\beta + t)\lambda},$$

$$= \frac{(\beta + t)^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-(\beta + t)\lambda},$$

which is itself a gamma density with parameters α and $(\beta + t)$. It represents the density for the distribution of failure rates in the population of survivors at time t. The relationship between G and G_t in this case is

$$\Lambda(t) = \frac{\beta}{\beta + t} \Lambda,$$

that is, $\Lambda(t)$ may be obtained from Λ by a change of scale.

II. STOCHASTIC ORDERING IN RESIDUAL MIXING DISTRIBUTIONS

A. INTRODUCTION

As mentioned in section I.C., the fact that the sequence of residual mixing distributions is decreasing in expectation leads to a conjecture of a compatible stochastic ordering of such a sequence. In this section the existence of such a stochastic ordering is demonstrated. It is also shown that the residual mixing distributions converge in distribution to the distribution degenerate at the least parameter value positively present in the mixture, with an attendant monotone convergence of raw moments. The latter result is one asserted by Aldrich and Morton [18].

The results are presented as applications of Propositions 2.1 and 2.2, to be developed in the next section. These propositions state properties of $E\{u[\Lambda(t)]\}$ for fairly general functions u. To indicate the relevance of these propositions, we first note that an expression of this form is involved in (1.3), taking $u[\Lambda(t)] = \Lambda(t)$. We next define an indicator function

$$I_\lambda(\Lambda) = \begin{cases} 0, & \Lambda \leq \lambda \\ 1, & \Lambda > \lambda \end{cases} \tag{2.1}$$

Using this indicator function, $\overline{G}_t$ may be expressed as

$$\overline{G}_t(\lambda) = \int_\lambda^\infty dG_t(x) = E\{I_\lambda[\Lambda(t)]\}, \tag{2.2}$$

another expression of the same form, with $u[\Lambda(t)] = I_\lambda[\Lambda(t)]$.

B. BASIC RESULTS

We shall develop in Propositions 2.1 and 2.2 of this section basic tools which should prove to be useful in a wide variety of applications. Several of these applications are given in section II.C.

Using (1.2) we find that

$$\begin{aligned} E\{u[\Lambda(t)]\} &= \int_0^\infty u(\lambda)dG_t(\lambda) \\ &= \int_0^\infty \frac{u(\lambda)e^{-\lambda t}}{E(e^{-\Lambda t})} dG(\lambda) \qquad (2.3) \\ &= \frac{E[u(\Lambda)e^{-\Lambda t}]}{E(e^{-\Lambda t})} . \end{aligned}$$

Thus, expectations of functions of $\Lambda(t)$ may be evaluated by simply taking expectations of functions of Λ. A straightforward application of the Monotone Convergence Theorem yields

$$\begin{aligned} \frac{d}{dt}\{E[u(\Lambda)e^{-\Lambda t}]\} &= E[u(\Lambda) \frac{d}{dt} e^{-\Lambda t}] \qquad (2.4) \\ &= -E[\Lambda u(\Lambda)e^{-\Lambda t}], \end{aligned}$$

for $\Lambda \geq 0$.

We next require the function u to be nonnegative and monotone to get

PROPOSITION 2.1. If Λ is a nonnegative random variable and u a nonnegative monotone increasing (decreasing) function, then

$$\phi_u(t) = \frac{E[u(\Lambda)e^{-\Lambda t}]}{E(e^{-\Lambda t})} = E\{u[\Lambda(t)]\}$$

is monotone decreasing (increasing) in t.

Proof: Using (2.4),

$$\begin{aligned}\frac{d}{dt}\phi_u(t) &= \frac{E(e^{-\Lambda t})E[-\Lambda u(\Lambda)e^{-\Lambda t}] - E[u(\Lambda)e^{-\Lambda t}]E[-\Lambda e^{-\Lambda t}]}{E(e^{-\Lambda t})^2} \\ &= -\frac{E[\Lambda u(\Lambda)e^{-\Lambda t}]}{E(e^{-\Lambda t})} + \frac{E[u(\Lambda)e^{-\Lambda t}]}{E(e^{-\Lambda t})}\left[\frac{E(\Lambda e^{-\Lambda t})}{E(e^{-\Lambda t})}\right] \\ &= E\{u[\Lambda(t)]\}E[\Lambda(t)] - E\{\Lambda(t)u[\Lambda(t)]\} \\ &= -\mathrm{COV}\{u[\Lambda(t)], \Lambda(t)\}.\end{aligned}$$

Then, by the property of the covariance of similarly/dissimilarly ordered functions of the same random variable,

(i) u increasing in $\Lambda(t) \Rightarrow \frac{d}{dt}\phi_u(t) \leq 0$, and

(ii) u decreasing in $\Lambda(t) \Rightarrow \frac{d}{dt}\phi_u(t) \geq 0$. □

We now need two lemmas regarding limits of expectations involving time t as a parameter.

LEMMA 2.1. Given u a nonnegative function,

$$\lim_{t\to\infty} E[I_\lambda(\Lambda)u(\Lambda)e^{-(\Lambda-\lambda)t}] = 0.$$

Proof: Note that $I_\lambda(\Lambda) = 1 \Leftrightarrow \Lambda - \lambda > 0 \Leftrightarrow e^{-(\Lambda-\lambda)t}\downarrow$ in t. An application of the Monotone Convergence Theorem yields

$$\lim_{t\to\infty} E[I_\lambda(\Lambda)u(\Lambda)e^{-(\Lambda-\lambda)t}] = 0.$$ □

LEMMA 2.2. For $\lambda > \lambda_o = \inf\{\lambda | \bar{G}(\lambda) < 1\}$,

$$\lim_{t\to\infty} E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] > 0.$$

Proof: Note that $(1-I_\lambda(\Lambda)) = 1 \Leftrightarrow \Lambda - \lambda \leq 0 \Leftrightarrow e^{-(\Lambda-\lambda)t} \geq 1$ for $t \geq 0$. Then, $E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] \geq E[1-I_\lambda(\Lambda)] = G(\lambda)$, and $\lim_{t\to\infty} E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] \geq G(\lambda) > 0$. □

If, in addition to the monotonicity requirement of Proposition 2.1, we also require u to be right-continuous at λ_o, we may find the limiting value of $\phi_u(t)$ as $t \to \infty$. We state this result in

PROPOSITION 2.2. Let Λ be a nonnegative random variable having cdf G, and u a nonnegative monotone increasing (decreasing) function. If u is right-continuous at $\lambda_o = \inf\{\lambda | \bar{G}(\lambda) < 1\}$, then $\phi_u(t)$ decreases (increases) monotonically to $u(\lambda_o)$ as $t \to \infty$.

Proof:

(i) If $u(\Lambda)\uparrow$ in Λ then $\phi_u(t) \geq u(\lambda_o)$. By Proposition 2.1, $\phi_u(t)\downarrow$ in t. Thus, $\lim_{t\to\infty} \phi_u(t)$ exists and lies in the interval $[u(\lambda_o),\infty)$. Choose $\lambda > \lambda_o$. Then,

$$\phi_u(t) = \frac{E[(1-I_\lambda(\Lambda))u(\Lambda)e^{-\Lambda t}] + E[I_\lambda(\Lambda)u(\Lambda)e^{-\Lambda t}]}{E[(1-I_\lambda(\Lambda))e^{-\Lambda t}] + E[I_\lambda(\Lambda)e^{-\Lambda t}]}$$

$$= \frac{E[(1-I_\lambda(\Lambda))u(\Lambda)e^{-(\Lambda-\lambda)t}] + E[I_\lambda(\Lambda)u(\Lambda)e^{-(\Lambda-\lambda)t}]}{E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] + E[I_\lambda(\Lambda)e^{-(\Lambda-\lambda)t}]},$$

Since $u(\Lambda)\uparrow$ in Λ,

$$\phi_u(t) \leq \frac{u(\lambda)E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] + E[I_\lambda(\Lambda)u(\Lambda)e^{-(\Lambda-\lambda)t}]}{E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] + E[I_\lambda(\Lambda)e^{-(\Lambda-\lambda)t}]},$$

$$\leq u(\lambda) + \frac{E[I_\lambda(\Lambda)u(\Lambda)e^{-(\Lambda-\lambda)t}]}{E[(1-I_\lambda(\Lambda))e^{-(\Lambda-\lambda)t}] + E[I_\lambda(\Lambda)e^{-(\Lambda-\lambda)t}]}, \tag{1}$$

Lemmas 2.1 and 2.2 may be applied to show that the last term in expression (1) goes to zero as $t \to \infty$. Therefore, $\lim_{t\to\infty} \phi_u(t) \leq u(\lambda)$. Let $\lambda \downarrow \lambda_o$ to get $\lim_{t\to\infty} \phi_u(t) \leq u(\lambda_o)$. then, $\phi_u(t) \downarrow u(\lambda_o)$ as $t \to \infty$.

(ii) The proof for u monotone decreasing is similar to that above. □

C. APPLICATIONS

The first application of the tools developed in the preceding section concerns the stochastic ordering of the sequence of residual mixing distributions and its limiting distribution, stated as

Theorem 2.1. If Λ is a nonnegative random variable with cdf G and $\lambda_o = \inf\{\lambda \mid \overline{G}(\lambda)<1\}$, then for a G-mixture of exponentials

a) G_t is a stochastically decreasing sequence in t, i.e.,

$$t_1 \leq t_2 \Rightarrow \overline{G}_{t_1}(\lambda) \geq \overline{G}_{t_2}(\lambda), \quad \text{for all } \lambda.$$

b) G_t converges in distribution to the distribution degenerate at λ_o, as $t \to \infty$.

Proof:

a) By (2.2),

$$\overline{G}_t(\lambda) = E\{I_\lambda[\Lambda(t)]\}.$$

Note that I_λ is monotone increasing in its argument. Application of Proposition 2.1 with $u[\Lambda(t)] = I_\lambda[\Lambda(t)]$ yields $\overline{G}_t(\lambda)\downarrow$ in t.

b) Application of Proposition 2.2 with $u[\Lambda(t)] = I_\lambda[\Lambda(t)]$ yields as $t \to \infty$,

$$\overline{G}_t(\lambda) \to I_\lambda(\lambda_o) = \begin{cases} 0, & \lambda_o \leq \lambda \\ 1, & \lambda_o > \lambda. \end{cases} \qquad \square$$

This result strengthens the statements which can be made about the population of survivors at time t. It was previously known that $\{G_t\}$ is a sequence decreasing in expectation. Now, in addition we may see that, given any value λ, the proportion of items in the population of survivors having a failure rate exceeding λ (indicated by $\overline{G}_t(\lambda)$) is decreasing over time. This relationship is clearly indicated in the examples of section I.D. by the monotonicity of the residual mixing probabilities in the two-point discrete mixture, and in the gamma mixture, by recalling that the relationship between Λ and $\Lambda(t)$ is

$$\Lambda(t) = \frac{\beta}{\beta + t}\,\Lambda.$$

Then, for $t_1 < t_2$, we have

$$\bar{G}_{t_1}(\lambda) = P[\Lambda(t_1)>\lambda] = P[\Lambda > \frac{\beta+t_1}{\beta}\lambda]$$

$$\geq P[\Lambda > \frac{\beta+t_2}{\beta}\lambda] = P[\Lambda(t_2)>\lambda] = \bar{G}_{t_2}(\lambda).$$

An immediate consequence of Proposition 2.2 is the well known result that a mixture of exponentials is DFR, with its failure rate approaching the least parameter value positively present in the mixture. We state this result as

Theorem 2.2. If Λ is a nonnegative random variable with cdf G and $\lambda_o = \inf\{\lambda | \bar{G}(\lambda)<1\}$, then for a G-mixture of exponentials the mixture failure rate function, $h(t)$, decreases monotonically to λ_o as $t \to \infty$.

Proof: As noted in (1.3),

$$h(t) = E[\Lambda(t)].$$

Application of Proposition 2.2 with $u[\Lambda(t)] = \Lambda(t)$ yields the desired result. □

This result may be seen to hold in the examples of section I.D. by noting the expressions found for $h(t)$ in (1.4) and (1.5).

We may also see the monotone convergence for the raw moments of the residual mixing distribution in

Theorem 2.3. For $p \geq 0$, $E[\Lambda(t)^p]$ decreases monotonically to λ_o^p .

Proof: Application of Proposition 2.2 with $u[\Lambda(t)] = \Lambda(t)^p$ yields the desired result. ▯

The variance of $\Lambda(t)$ is of interest, since it has an interpretation as the negative of the slope of the mixture failure rate function [O'Bar, 19]. We state the following as

Corollary 1. The limit as $t \to \infty$ of $VAR[\Lambda(t)]$ is zero.

Proof: We express the variance of $\Lambda(t)$ as

$$VAR[\Lambda(t)] = E[\Lambda(t)^2] - E^2[\Lambda(t)] \tag{2.5}$$

Then, by Theorem 2.3,

$$\lim_{t\to\infty} VAR[\Lambda(t)] = \lambda_o^2 - \lambda_o^2 = 0. \qquad ▯$$

Expression (2.5) leads us to

Corollary 2. On the interval $(0,\infty)$, $VAR[\Lambda(t)]$ is of bounded variation.

Proof: From the power series expansion for $e^{\Lambda t}$, we have that

$$\Lambda^n < \frac{n!}{t^n} e^{\Lambda t} ,$$

for n a nonnegative integer. Thus

$$E[\Lambda(t)^n] = \frac{E[\Lambda^n e^{-\Lambda t}]}{E(e^{-\Lambda t})} < \frac{n!}{t^n} \frac{E[e^{\Lambda t} e^{-\Lambda t}]}{E(e^{-\Lambda t})}$$

$$< \frac{n!}{t^n E(e^{-\Lambda t})} , \text{ for } t > 0.$$

Then, from expression (2.5) and Theorem 2.3, since $VAR[\Lambda(t)]$ is expressed as the difference of two real-valued nonnegative monotone functions for $t > 0$, it is of bounded variation on $(0, \infty)$. $\square$

III. ORDERINGS OVER TIME AMONG DISTINCT G-MIXTURES OF EXPONENTIALS

A. INTRODUCTION

Chapter II dealt with the behavior over time of the sequence of residual mixing distributions $\{G_t\}$ corresponding to a single given mixing distribution G.

The applications of the model (1.1) in a reliability context prompt questions regarding orderings among distinct initial mixing distributions G_i or the creation of some other such ordering between their corresponding residual mixing distributions $G_{i,t}$ over time. Specifically, if the initial mixing distribution can be altered, what conditions upon the alteration can guarantee that the altered mixture would be in some sense better uniformly over time? This chapter contains some results in this direction.

B. BASIC RESULTS

We examine several orderings, their properties and their interrelationships in sections III.B.1 through III.B.4.

1. Stochastic Ordering

We take the usual definition for stochastic ordering as employed in Chapter II; that is, two random variables are stochastically ordered if and only if their survival functions exhibit that ordering,

$$\Lambda_1 \leq^{st} \Lambda_2 \Leftrightarrow \overline{G}_1(\lambda) \leq \overline{G}_2(\lambda), \quad \text{for all} \quad \lambda.$$

Another useful characterization of stochastic ordering states that $\Lambda_1 \leq^{st} \Lambda_2 \Leftrightarrow E[u(\Lambda_1)] \leq E[u(\Lambda_2)]$, for all u monotone increasing. This characterization follows directly from a lemma by Lehmann [21, p. 73] which states that $\Lambda_1 \leq^{st} \Lambda_2$ if and only if there exist two monotone increasing functions f_1 and f_2 and a random variable V, with $f_1(v) \leq f_2(v)$ for all v and $\Lambda_1 =^{st} f_1(V)$, $\Lambda_2 =^{st} f_2(V)$. That $E[u(\Lambda_1)] \leq E[u(\Lambda_2)]$ is a necessary condition follows from noting that $E[u(\Lambda_1)] = E[uf_1(V)] \leq E[uf_2(V)] = E[u(\Lambda_2)]$. That it is sufficient also may be seen by taking

$$u(\Lambda) = I_\lambda(\Lambda) = \begin{cases} 1, & \text{if } \Lambda > \lambda \\ 0, & \text{if } \Lambda \leq \lambda. \end{cases}$$

An equivalent characterization is $\Lambda_1 \leq^{st} \Lambda_2 \Leftrightarrow E[u(\Lambda_1)] \geq E[u(\Lambda_2)]$, for all u monotone decreasing.

2. Failure Rate Ordering

We define a _failure rate ordering_ between two random variables by

$$\Lambda_1 \leq^{fr} \Lambda_2 \Leftrightarrow E_p[I_{u,v}(\Lambda_2,\Lambda_1) - I_{u,v}(\Lambda_1,\Lambda_2)] \geq 0, \tag{3.1}$$

where for $u > v$ the indicator function $I_{u,v}$ is defined by

$$I_{u,v}(x,y) = \begin{cases} 1, & \text{if } x > u,\ y > v \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

and where E_p indicates an expected value with respect to

the product measure of Λ_1 and Λ_2 for Λ_1 and Λ_2 independent. We show the equivalence of the definition above and an ordering of failure rate functions where they exist in

Lemma 3.1. Given two random variables, Λ_1 and Λ_2, with failure rate functions h_1 and h_2 respectively, then

$$h_1(\lambda) \geq h_2(\lambda) \quad \text{for all } \lambda \iff \Lambda_1 \leq^{fr} \Lambda_2.$$

Proof: ($\Rightarrow$) Suppose $h_1(\lambda) \geq h_2(\lambda)$ for all λ. Then, for $u > v$,

$$\int_v^u h_1(s)ds \geq \int_v^u h_2(s)ds$$

$$\exp - \int_v^u h_1(s)ds \leq \exp - \int_v^u h_2(s)ds$$

$$\frac{\overline{G}_1(u)}{\overline{G}_1(v)} \leq \frac{\overline{G}_2(u)}{\overline{G}_2(v)}$$

$$\overline{G}_2(u)\overline{G}_1(v) - \overline{G}_2(v)\overline{G}_1(u) \geq 0$$

$$P(\Lambda_2 > u)P(\Lambda_1 > v) - P(\Lambda_2 > v)P(\Lambda_1 > u) \geq 0$$

$$E_p[I_{u,v}(\Lambda_2,\Lambda_1) - I_{u,v}(\Lambda_1,\Lambda_2)] \geq 0$$

($\Leftarrow$) Reverse the above steps. □

3. Likelihood Ratio Ordering

We define a likelihood ratio ordering between two random variables by $\Lambda_1 \overset{\ell r}{\leq} \Lambda_2 \Leftrightarrow E_p[\phi(\Lambda_2,\Lambda_1)] \geq 0$ for all functions ϕ such that $\phi(x,y) = -\phi(y,x)$, and $\phi(x,y) \geq 0$ for $x \geq y$.

This ordering is equivalent, in the case where the random variables have densities, to the assertion that Λ_1 and Λ_2 belong to a family of densities having monotone likelihood ratio [21]. Other characterizations of this ordering in the continuous case are given in [22] and [23]. These state that Λ_1 and Λ_2 are said to be totally positive of order two, or Polya type II if and only if

$$\begin{vmatrix} g_1(\lambda_1) & g_1(\lambda_2) \\ g_2(\lambda_1) & g_2(\lambda_2) \end{vmatrix} \geq 0, \text{ for } \lambda_1 < \lambda_2.$$

Equivalently, that

$$\frac{g_1(\lambda_2)}{g_1(\lambda_1)} \leq \frac{g_2(\lambda_2)}{g_2(\lambda_1)}, \text{ for } \lambda_1 < \lambda_2. \tag{3.3}$$

We show the equivalence of the definition given for likelihood ratio ordering and equation (3.3) in the continuous case in

Lemma 3.2. Given two random variables, Λ_1 and Λ_2, with densities g_1 and g_2 respectively, then

$$\frac{g_1(\lambda_2)}{g_1(\lambda_1)} \leq \frac{g_2(\lambda_2)}{g_2(\lambda_1)}, \ \lambda_1 < \lambda_2 \Leftrightarrow E_p[\phi(\Lambda_2,\Lambda_1)] \geq 0,$$

for all functions ϕ such that $\phi(x,y) = -\phi(y,x)$ and $\phi(x,y) \geq 0$, for $x \geq y$.

Proof: $(\Rightarrow)$ $E_p[\phi(\Lambda_2,\Lambda_1)] = \int_0^\infty \int_0^\infty \phi(x,y)g_2(x)g_1(y)dxdy$

$$= \int_0^\infty \int_0^y \phi(x,y)g_2(x)g_1(y)dxdy$$

$$+ \int_0^\infty \int_y^\infty \phi(x,y)g_2(x)g_1(y)dxdy$$

$$= \int_0^\infty \int_x^\infty \phi(x,y)g_2(x)g_1(y)dydx$$

$$+ \int_0^\infty \int_y^\infty \phi(x,y)g_2(x)g_1(y)dxdy$$

We make changes of variables of $x = v$, $y = u$ in the first integral and $y = v$, $x = u$ in the second integral to get

$$E_p[\phi(\Lambda_2,\Lambda_1)] = \int_0^\infty \int_v^\infty \phi(v,u)g_2(v)g_1(u)dudv$$

$$+ \int_0^\infty \int_v^\infty \phi(u,v)g_2(u)g_1(v)dudv$$

$$= -\int_0^\infty \int_v^\infty \phi(u,v)g_2(v)g_1(u)dudv$$

$$+ \int_0^\infty \int_v^\infty \phi(u,v)g_2(u)g_1(v)dudv$$

$$E_p[\phi(\Lambda_2,\Lambda_1)] = \int_0^\infty \int_v^\infty \phi(u,v)[g_2(u)g_1(v) - g_2(v)g_1(u)]\,du\,dv$$

Since $u \geq v$ over the range of integration, the integrand is nonnegative by hypothesis and property two of the function ϕ.

($\Leftarrow$) Let $0 \leq z_1 < z_2$. Choose Δz such that $0 < \Delta z < (z_2 - z_1)$. Let $I_i = (z_i, z_i + \Delta z]$ for $i = 1,2$. Define

$$\phi(x,y) = \begin{cases} +1, & \text{for } x \in I_2, \quad y \in I_1 \\ -1, & \text{for } x \in I_1, \quad y \in I_2 \\ 0, & \text{otherwise} \end{cases}$$

The function $\phi(\Lambda_2,\Lambda_1)$ is shown in Figure 3.1.

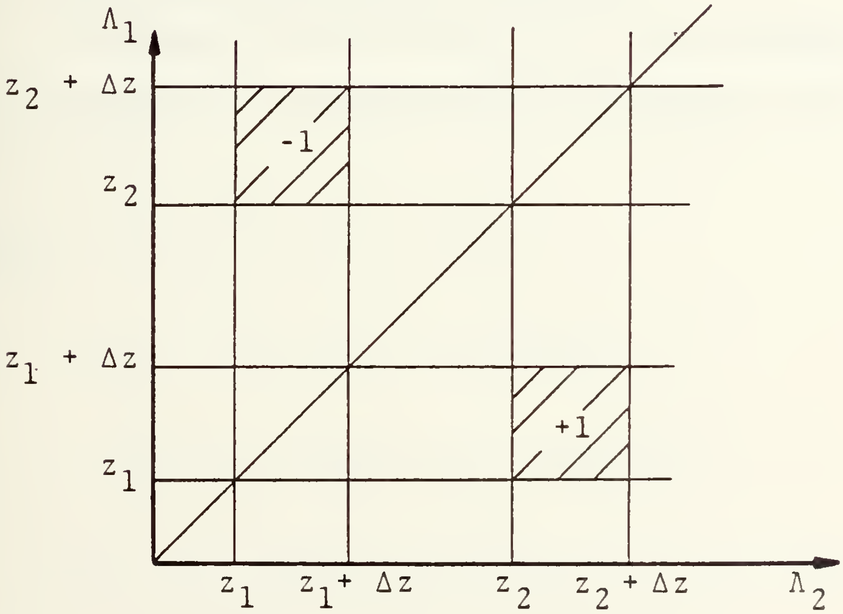

Figure 3.1.

Then, $E_p[\phi(\Lambda_2,\Lambda_1)] = P(\Lambda_2 \varepsilon I_2)P(\Lambda_1 \varepsilon I_1) - P(\Lambda_2 \varepsilon I_1)P(\Lambda_1 \varepsilon I_2) \geq 0$, which yields

$$\frac{P(\Lambda_2 \varepsilon I_2)P(\Lambda_1 \varepsilon I_1)}{\Delta z^2} - \frac{P(\Lambda_2 \varepsilon I_1)P(\Lambda_1 \varepsilon I_2)}{\Delta z^2} \geq 0$$

$$\frac{G_2(z_2 + \Delta z) - G_2(z_2)}{\Delta z} \; \frac{G_1(z_1 + \Delta z) - G_1(z_1)}{\Delta z}$$

$$- \frac{G_2(z_1 + \Delta z) - G_2(z_1)}{\Delta z} \; \frac{G_1(z_2 + \Delta z) - G_1(z_2)}{\Delta z} \geq 0$$

Let $\Delta z \to 0$ to get $g_2(z_2)g_1(z_1) - g_2(z_1)g_1(z_2) \geq 0$, which completes the proof. $\square$

4. Relationships Among Orderings

Likelihood ratio ordering is the strongest of the three orderings. Its implication of failure rate ordering may be seen in

Lemma 3.3. $\Lambda_1 \leq^{\ell r} \Lambda_2 \Rightarrow \Lambda_1 \leq^{fr} \Lambda_2$.

Proof: Follows directly by defining

$$\phi(x,y) = I_{u,v}(x,y) - I_{u,v}(y,x)$$

for the indicator function defined by (3.2), and noting that $\phi(x,y) = -\phi(y,x)$ and $\phi(x,y) \geq 0$ for $x \geq y$. $\square$

That the reverse implication does not hold may be easily seen by letting $n = 3$, $j = 2$ and $k = 3$ in the counter-example of section III.C.2.a.

We get a direct verification of the well-known result that failure rate ordering implies stochastic ordering in

Lemma 3.4. $\Lambda_1 \leq^{fr} \Lambda_2 \Rightarrow \Lambda_1 \leq^{st} \Lambda_2$.

Proof: Follows directly by taking v = 0 in (3.1) $\square$

Letting $u(\Lambda) = \Lambda$ in the characterization for stochastically ordered random variables of section III.B.1. with respect to monotone increasing u yields the familiar result that

$$\Lambda_1 \leq^{st} \Lambda_2 \Rightarrow E(\Lambda_1) \leq E(\Lambda_2).$$

C. APPLICATIONS TO MIXTURES OF EXPONENTIALS

1. Ordering Inheritance Hypotheses

We first formulate hypotheses asserting inheritance properties between orderings among initial mixing distributions and their respective residual mixing distributions. They may be stated as

H1: $E(\Lambda_1) \leq E(\Lambda_2) \Rightarrow E[\Lambda_1(t)] \leq E[\Lambda_2(t)],\ t \geq 0.$

H2: $\Lambda_1 \leq^{st} \Lambda_2 \Rightarrow \Lambda_1(t) \leq^{st} \Lambda_2(t),\ t \geq 0.$

H3: $\Lambda_1 <^{fr} \Lambda_2 \Rightarrow \Lambda_1(t) \leq^{fr} \Lambda_2(t),\ t \geq 0.$

H4: $\Lambda_1 \leq^{lr} \Lambda_2 \Rightarrow \Lambda_1(t) \leq^{lr} \Lambda_2(t),\ t \geq 0.$

We introduce the following set identifications:

$A = \{(\Lambda_1,\Lambda_2) \mid E(\Lambda_1) \leq E(\Lambda_2)\}$

$B = \{(\Lambda_1,\Lambda_2) \mid E[\Lambda_1(t)] \leq E[\Lambda_2(t)],\ t \geq 0\}$

$$C = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1 \leq^{st} \Lambda_2\}$$

$$D = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1(t) \leq^{st} \Lambda_2(t),\ t \geq 0\}$$

$$E = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1 \leq^{fr} \Lambda_2\}$$

$$F = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1(t) \leq^{fr} \Lambda_2(t),\ t \geq 0\}$$

$$G = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1 \leq^{lr} \Lambda_2\}$$

$$H = \{(\Lambda_1,\Lambda_2) \mid \Lambda_1(t) \leq^{lr} \Lambda_2(t),\ t \geq 0\}$$

We can now restate the hypotheses as

H1: $A \subseteq B$

H2: $C \subseteq D$

H3: $E \subseteq F$

H4: $G \subseteq H$

We know, by letting $t \to 0$, that $B \subseteq A$, $D \subseteq C$ and $F \subseteq E$. Thus, hypotheses H1-H3 are actually postulating equivalences. We also know from the relationships given in section III.B.4. that $G \subseteq E \subseteq C \subseteq A$ and $H \subseteq F \subseteq D \subseteq B$.

2. Counter-examples to Hypotheses

a. Hypotheses H1 and H2

We first state an additional hypothesis:

H5: $\Lambda_1 \leq^{st} \Lambda_2 \Rightarrow E[\Lambda_1(t)] \leq E[\Lambda_2(t)],\ t \geq 0.$ $(C \subseteq B)$

From section III.B.4. we know that $C \subseteq A$ and that $D \subseteq B$. If H5 is false, then $C \cap \tilde{B}$ is not empty. That would imply that

neither $A \cap \tilde{B}$ nor $C \cap \tilde{D}$ is empty. Thus, hypotheses H1 and H2 would be disproved.

H5 is shown to be false as follows: Let $\Lambda_2 = \lambda_i$ with probability p_i, $i = 1,\ldots,n$, where $\lambda_1 < \lambda_2 <..<\lambda_n$. Then, for $1 \leq j < k \leq n$, let

$$\Lambda_1 = \begin{cases} \lambda_j & \text{with probability } p_j + \varepsilon \\ \lambda_k & \text{with probability } p_k - \varepsilon \\ \lambda_i & \text{with probability } p_i,\ i \neq j,k \end{cases}$$

By inspection, $\Lambda_1 \overset{st}{\leq} \Lambda_2$. Let $I = \{i \mid i \in \{1,2,\ldots,n\};\ i \neq j,k\}$. Then we can write, summing from 1 to n unless otherwise noted,[1]

$$E[\Lambda_2(t)] = h_2(t) = \frac{\Sigma p_i \lambda_i e^{-\lambda_i t}}{\Sigma p_i e^{-\lambda_i t}}$$

$$E[\Lambda_1(t)] = h_1(t) = \frac{\sum_I p_i \lambda_i e^{-\lambda_i t} + (p_j + \varepsilon)\lambda_j e^{-\lambda_j t} + (p_k - \varepsilon)\lambda_k e^{-\lambda_k t}}{\sum_I p_i e^{-\lambda_i t} + (p_j + \varepsilon) e^{-\lambda_j t} + (p_k - \varepsilon) e^{-\lambda_k t}}$$

$$= \frac{\Sigma p_i \lambda_i e^{-\lambda_i t} + \varepsilon(\lambda_j e^{-\lambda_j t} - \lambda_k e^{-k t})}{\Sigma p_i e^{-\lambda_i t} + \varepsilon(e^{-\lambda_j t} - e^{-\lambda_k t})}$$

Assume that $E[\Lambda_1(t)] \leq E[\Lambda_2(t)]$. Then we have

$$\frac{\Sigma p_i \lambda_i e^{-\lambda_i t} + \varepsilon(\lambda_j e^{-\lambda_j t} - \lambda_k e^{-\lambda_k t})}{\Sigma p_i e^{-\lambda_i t} + \varepsilon(e^{-\lambda_j t} - e^{-\lambda_k t})} \leq \frac{\Sigma p_i \lambda_i e^{-\lambda_i t}}{\Sigma p_i e^{-\lambda_i t}}$$

---

[1] Here, $h_1(t)$ and $h_2(t)$ are the failure rate functions for T_1 and T_2 respectively.

$$\Sigma p_i e^{-\lambda_i t}[\Sigma p_i \lambda_i e^{-\lambda_i t} + \varepsilon(\lambda_j e^{-\lambda_j t} - \lambda_k e^{-\lambda_k t})]$$

$$\leq \Sigma p_i \lambda_i e^{-\lambda_i t}[\Sigma p_i e^{-\lambda_j t} + \varepsilon(e^{-\lambda_j t} - e^{-\lambda_k t})]$$

$$\frac{\lambda_j e^{-\lambda_j t} - \lambda_k e^{-\lambda_k t}}{e^{-\lambda_j t} - e^{-\lambda_k t}} \leq h_2(t)$$

$$\frac{\lambda_j - \lambda_k e^{-(\lambda_k - \lambda_j)t}}{1 - e^{-(\lambda_k - \lambda_j)t}} \leq h_2(t) \tag{3.4}$$

Taking the limit as t → ∞ in expression (3.4), $h_2(t) \rightarrow \lambda_1$ and according to Theorem 2.2, we have $\lambda_j \leq \lambda_1$. Therefore, $j = 1$ is a necessary condition, that is, for $j \neq 1$, the assumption that $E[\Lambda_1(t)] \leq E[\Lambda_2(t)]$ leads to a contradiction. Then, hypotheses H1, H2 and H5 are false.

b. Hypothesis H3

We first state an additional hypothesis:

H6: $\Lambda_1 \leq^{\delta r} \Lambda_2 \Rightarrow E[\Lambda_1(t)] \leq E[\Lambda_2(t)],\ t \geq 0.$ $(E \subseteq B)$

From section III.B.4. we know that $F \subseteq D \subseteq B$. If H6 is false, then $E \cap \tilde{B}$ is not empty which implies that $E \cap \tilde{F}$ is not empty, disproving H3.

H6 is shown to be false by taking $k = n$ and $j \neq 1$ in the counter-example of section III.C.2.a. Using the definition (3.1), it is easily verified that $\Lambda_1 \leq^{\delta r} \Lambda_2$. From that counter-example, however, we know that $j = 1$ is a necessary condition for B. Therefore, H6 is false, disproving H3.

3. Proof of Hypothesis H4

We can go beyond hypothesis H4 and show an equivalence between likelihood ratio orderings in the initial and residual mixing distributions in

Proposition 3.1. $\Lambda_1 \leq^{\ell r} \Lambda_2 \Leftrightarrow \Lambda_1(t) \leq^{\ell r} \Lambda_a(t)$, $t \geq 0$.

Proof: ($\Rightarrow$) From equation (2.3) we know that for a function u,

$$E\{u[\Lambda(t)]\} = \frac{E[u(\Lambda)e^{-\lambda t}]}{E(e^{-\Lambda t})}$$

Similarly,

$$E_p\{u[\Lambda_2(t),\Lambda_1(t)]\} = \frac{E_p[u(\Lambda_2,\Lambda_1)e^{-(\Lambda_1+\Lambda_2)t}]}{E_p[e^{-(\Lambda_1+\Lambda_2)t}]}$$

Choose a function ϕ such that $\phi(x,y) = -\phi(y,x)$ and $\phi(x,y) \geq 0$, for $x \geq y$. Then

$$E_p\{\phi[\Lambda_2(t),\Lambda_1(t)]\} = \frac{E_p[\phi(\Lambda_2,\Lambda_1)e^{-(\Lambda_1+\Lambda_2)t}]}{E_p[e^{-(\Lambda_1+\Lambda_2)t}]}$$

$$= \frac{E_p[\psi_t(\Lambda_2,\Lambda_1)]}{E_p[e^{-(\Lambda_1+\Lambda_2)t}]},$$

where $\psi_t(x,y) = \phi(x,y)e^{-(x+y)t}$. Note that $\psi_t(x,y) = -\psi_t(y,x)$ and $\psi_t(x,y) \geq 0$ for $x \geq y$. By hypothesis, then, $E_p[\Psi_t(\Lambda_2,\Lambda_1)] \geq 0$, and $\Lambda_1(t) \leq^{\ell r} \Lambda_2(t)$.

$(\Leftarrow)$ $$E_p\{\phi[\Lambda_2(t),\Lambda_1(t)]\} \geq 0$$

$$\frac{E_p[\phi(\Lambda_2,\Lambda_1)e^{-(\Lambda_1+\Lambda_2)t}]}{E_p[e^{-(\Lambda_1+\Lambda_2)t}]} \geq 0$$

Then, $E_p[\phi(\Lambda_2,\Lambda_1)] \geq 0$, and $\Lambda_1 \overset{\ell r}{\leq} \Lambda_2$. $\square$

With the proof of Proposition 3.1, then, we have established that, of the orderings considered, only likelihood ratio ordering between two initial mixing distributions is sufficient to guarantee an improvement in the mixture failure rate function for any time $t \geq 0$. This would seem to be an extremely strong requirement; however, according to Karlin [23], most of the more common distributions belong to this class. The gamma distribution, in particular, is a member. We can show this by noting for the gamma distribution, for $u > v$,

$$\frac{g(u)}{g(v)} = \frac{u^{\alpha-1}e^{-\beta u}}{v^{\alpha-1}e^{-\beta v}} = \left(\frac{u}{v}\right)^{\alpha-1} e^{-\beta(u-v)}.$$

We note that this expression is increasing in α and decreasing in β. So, whether we are changing α or β, holding the other fixed, the result will be two distributions which are likelihood ratio ordered. To see that such an alteration would, in fact, lead to an ordering of the mixture failure rate function over time, recall expression (1.5) which showed

$$h(t) = \frac{\alpha}{\beta + t} .$$

4. Shifting Mass from One Population to Another

A common situation is one in which an item which fails according to an exponential life distribution is obtained from several sources or populations having different failure rates. The survival function for the mixture then becomes

$$\bar{F}(t) = \sum_{i=1}^{n} p_i e^{-\lambda_i t}$$

for n populations, each having a failure rate λ_i and comprising a part of the mixture p_i. An obvious way to improve the mixture would be to shift mass from one population to another having a lower failure rate. From the counter-example given in section III.C.2.a, we know that in order to guarantee an improvement in the failure rate of the mixture uniformly over time (that is, for a shift from Λ_2 to Λ_1, we want $E[\Lambda_1(t)] \leq E[\Lambda_2(t)]$, $t \geq 0$) the population receiving the increase in mass must be that one having the minimum failure rate. That is, shifting mass to the most reliable population in the mixture is a necessary condition. Likelihood ratio ordering among initial mixing distributions was shown in section III.C.3. to be a sufficient condition for such a uniform improvement in the failure rate.

Recall the statement of the example of section III.C.2.a. As noted above, we know that j = 1. Through a procedure analagous to that in the proof of Lemma 3.2, using sums instead of integrals, it is easily seen that in this case for $\Lambda_1 \overset{\ell r}{\leq} \Lambda_2$ we have, for i > m,

$$\frac{P(\Lambda_1 = \lambda_i)}{P(\Lambda_1 = \lambda_m)} \leq \frac{P(\Lambda_2 = \lambda_i)}{P(\Lambda_2 = \lambda_m)} .$$

Suppose $k \neq n$. Let $m = k$ and $i = n$ to get

$$\frac{P(\Lambda_1 = \lambda_n)}{P(\Lambda_1 = \lambda_k)} = \frac{p_n}{p_k - \varepsilon} > \frac{p_n}{p_k} = \frac{P(\Lambda_2 = \lambda_n)}{P(\Lambda_2 = \lambda_k)} ,$$

and $\Lambda_1 \not\leq^{\ell r} \Lambda_2$. Then $k = n$ and shifting mass from the worst population to the best population is sufficient for an improvement in the failure rate of the mixture uniformly over time.

We note here that the intuitively appealing approach of simply shifting mass from a poorer population to a better one will not lead to a lower mixture failure rate function over all time. Unless the shift is made from the worst to the best, there will be some time at which the two mixture failure rate functions will cross.

5. Likelihood Ratio Ordering of T_1 and T_2

We show a likelihood ratio ordering among initial mixing distributions to imply a likelihood ratio ordering of the opposite sense in the times to failure of the mixture, T_1 and T_2, in

Proposition 3.2. $\Lambda_1 \leq^{\ell r} \Lambda_2 \Rightarrow T_1 \geq^{\ell r} T_2$.

Proof: Suppose $\Lambda_1 \leq^{\ell r} \Lambda_2$. Define the function

$$\phi(x,y) = \begin{cases} xy(e^{-(yt_2 + xt_1)} - e^{-(yt_1 + xt_2)}), & t_1 < t_2 \\ 0, \quad t_1 \geq t_2 \end{cases}$$

We note that $\phi(x,y) = -\phi(y,x)$ and $\phi(x,y) \geq 0$ for $x \geq y \geq 0$. By hypothesis, $E_p[\phi(\Lambda_2,\Lambda_1)] \geq 0$. Then, for $t_1 < t_2$,

$$E_p\{\Lambda_1\Lambda_2[e^{-(\Lambda_1 t_2 + \Lambda_2 t_1)} - e^{-(\Lambda_1 t_1 + \Lambda_2 t_2)}]\} \geq 0$$

$$E_p[\Lambda_1\Lambda_2 e^{-(\Lambda_1 t_2 + \Lambda_2 t_1)}] \geq E_p[\Lambda_1\Lambda_2 e^{-(\Lambda_1 t_1 + \Lambda_2 t_2)}]$$

$$E(\Lambda_1 e^{-\Lambda_1 t_2})E(\Lambda_2 e^{-\Lambda_2 t_1}) \geq E(\Lambda_1 e^{-\Lambda_1 t_1})E(\Lambda_2 e^{-\Lambda_2 t_2})$$

$$\frac{E(\Lambda_2 e^{-\Lambda_2 t_2})}{E(\Lambda_2 e^{-\Lambda_2 t_1})} \leq \frac{E(\Lambda_1 e^{-\Lambda_1 t_2})}{E(\Lambda_1 e^{-\Lambda_1 t_1})}\ , \quad t_1 < t_2\ ,$$

$$\frac{f_2(t_2)}{f_2(t_1)} \leq \frac{f_1(t_2)}{f_1(t_1)}\ , \quad t_1 < t_2$$

that is, $T_1 \geq^{\ell r} T_2$. □

6. Mean Residual Lifetime Ordering

The *mean residual lifetime* at time t corresponding to a distribution F is

$$\mu(t) = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)\,dx = E(t - t \mid T > t).$$

For $T_1 \leq^{\ell r} T_2$, we use equation (3.3) to write

$$\frac{\overline{F}_1(x)}{\overline{F}_1(t)} \leq \frac{\overline{F}_2(x)}{\overline{F}_2(t)}\ , \quad x \geq t$$

Then,

$$\mu_1(t) = \int_t^\infty \frac{\overline{F}_1(x)}{\overline{F}_1(t)}\, dx \leq \int_t^\infty \frac{\overline{F}_2(x)}{\overline{F}_2(t)}\, dx = \mu_2(t),$$

so, if $T_1 \leq^{\ell r} T_2$ then $\mu_1(t) \leq \mu_2(t)$, $t \geq 0$.

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